

## Lecture 23 (April 28, 2016)

Big picture of lecture 23 & 24:

Goal: Approximation of solutions of  $\dot{x} = f(t, x)$

Two distinct approximation methods

asymptotic methods ✓  
numerical methods (e.g. RK)

### ① Perturbation method

How to approximate the solutions of

$$\dot{x} = f(t, x, \varepsilon), \quad x(t_0) = \eta(\varepsilon) \quad \textcircled{X}$$

where  $f$  and  $\eta$  depend on the parameter  $\varepsilon$  (perturbation) smoothly?

Existence and uniqueness of solutions of  $\textcircled{X}$ :

In chapter 3, Thm 3.5, we saw that if  $\dot{x} = f(t, x, 0), \quad x(t_0) = \eta_0$  has a unique solution  $x_0(t)$  defined on  $I$ , then  $\textcircled{X}$  also has a unique solution defined on  $I$  that stay close to  $x_0(t)$ .

The goal of perturbation method is to exploit the "smallness" of the perturbation parameter  $\varepsilon$  to construct approximate solutions that are valid for sufficiently small  $\varepsilon$ .

The idea of perturbation method is to approximate the solution of  $\textcircled{X}$  by its finite Taylor expansion.

→ Periodic Perturbation  $\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon) \quad (\star\star\star)$

O: e.s. of  $\dot{x} = f(x) \xrightarrow{\text{Chap 9}}$  the solutions of  $(\star\star\star)$  are ultimately bounded

Question:  $g$ : T-periodic. Is there any limit cycle close to the origin?

$$② \text{ averaging} \quad (\star\star) \quad \dot{x} = \varepsilon f(t, x, \varepsilon) \quad \text{non-autonomous}$$

$f$  could be  $T$ -periodic or non-periodic.

The idea is to approximate the solution of  $(\star\star)$  by the solution of

$$\dot{x} = \varepsilon f_{av}(x) \quad \text{where } f_{av}(x) = \frac{1}{T} \int_0^T f(\tau, x, 0) d\tau \quad (\text{autonomous system})$$

when  $f$  is periodic (which we discuss here); ~~or by the solution of~~

or by the solution of

$$\dot{x} = \varepsilon f_{av}(x) \quad \text{where} \quad f_{av}(x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_t^{t+T} f(\tau, x, 0) d\tau$$

when  $f$  is not periodic.

(we don't discuss this today).

### ③ Singular Perturbation

Discontinuous dependence of system on the perturbation parameter  $\varepsilon$ .

standard singular perturbation model

$$(\star\star\star) \quad \dot{x} = f(t, x, z, \varepsilon), \quad x(t_0) = \xi(\varepsilon)$$

$$\varepsilon \dot{z} = g(t, x, z, \varepsilon) \quad z(t_0) = \eta(\varepsilon)$$

The discontinuity of solutions caused by singular perturbations can be avoided if analyzed in separate time scales.

The idea is to approximate the solutions of  $(\star\star\star)$  by the solutions of a slow system ( $\dot{x} = f(t, x, h(t, x, 0))$  where  $z = h(t, x)$  is the solution of  $g(t, x, z, 0) = 0$ ) and a fast system ( $\frac{dy}{d\tau} = g(t_0, \xi(0), h(t_0, \xi(0)), 0)$ ).

## The Perturbation method

$$\dot{x} = f(t, x, \epsilon) \quad f: [t_0, t_1] \times D \times [-\epsilon_0, \epsilon_0] \rightarrow \mathbb{R}^n, \quad D \subseteq \mathbb{R}^m \text{ sufficiently smooth}$$

$$x(t_0) = \eta(\epsilon) \quad \eta: \text{smooth}$$

Order of magnitude notation:

$\delta_1(\epsilon) = O(\delta_2(\epsilon))$  if  $\exists$  Positive constants  $K$  and  $c$  s.t.

$$|\delta_1(\epsilon)| \leq K |\delta_2(\epsilon)|, \quad \forall \epsilon < c$$

Example. For "sufficiently small"  $\epsilon$ , an  $O(\epsilon^n)$  term will be smaller than  $O(\epsilon^m)$  term for  $n > m$ .

Nominal system:  $\epsilon=0 \quad \dot{x} = f(t, x, 0)$

$$x(t_0) = \eta_0 = \eta(0)$$

Suppose:

- $x_0(t)$  is a unique solution on  $[t_0, t_1]$
- $x_0(t) \in D, \quad \forall t \in [t_0, t_1]$
- $f$  cont in  $(t, x, \epsilon)$ , locally Lipschitz in  $(x, \epsilon)$ , uniformly in  $t$
- $\eta$ : locally Lipschitz in  $\epsilon$

continuity of solution

$\exists 0 < \epsilon_1 \leq \epsilon_0$  s.t.  $\forall \epsilon \leq \epsilon_1$ , original system has a unique solution on  $[t_0, t_1]$  and  $\exists$  a positive constant  $K$  s.t.

$$\|x(t, \epsilon) - x_0(t)\| \leq K\epsilon \iff \|x(t, \epsilon) - x_0(t)\| = O(\epsilon)$$

$\forall \epsilon < \epsilon_1, \quad \forall t \in [t_0, t_1]$

Higher order approximation:  $x(t, \epsilon) = \sum_{k=0}^{N-1} x_k(t) \epsilon^k + \epsilon^N R_x(t, \epsilon)$

If  $R_N(t, \epsilon)$  is well-defined & bounded on  $[t_0, t_1]$ , then

$$\|x(t, \epsilon) - \sum_{k=0}^{N-1} x_k(t) \epsilon^k\| = O(\epsilon^N)$$

Let  $\eta(\epsilon) = \sum_{k=0}^{N-1} \eta_k \epsilon^k + \epsilon^N R_\eta(\epsilon) \Rightarrow x_k(t_0) = \eta_k, k = \{1, \dots, N-1\}$   
 (since  $x(t_0, \epsilon) = \eta(\epsilon)$ ).

$$\begin{aligned}\dot{x} &= \sum_{k=0}^{N-1} \dot{x}_k(t) \epsilon^k + \epsilon^N \dot{R}_x(t, \epsilon) = f(t, x(t, \epsilon), \epsilon) := h(t, \epsilon) \\ &= \sum_{k=0}^{N-1} h_k(t) \epsilon^k + \epsilon^N R_h(t, \epsilon)\end{aligned}$$

$h_k(t)$ : functions of  $x_k$ 's

$$h_0(t) = f(t, x_0(t, \epsilon), \epsilon) \Big|_{\epsilon=0} = f(t, x_0(t), 0)$$

$$\begin{aligned}h_1(t) &= \frac{\partial}{\partial \epsilon} f(t, x_0(t, \epsilon), \epsilon) \Big|_{\epsilon=0} = \left( \frac{\partial f}{\partial x} \frac{\partial x}{\partial \epsilon} + \frac{\partial f}{\partial \epsilon} \right) \Big|_{\epsilon=0} \\ &= \frac{\partial f}{\partial x}(t, x_0(t), 0) x_1(t) + \frac{\partial f}{\partial \epsilon}(t, x_0(t), 0)\end{aligned}$$

etc.

Equate coefficients of like powers of  $\epsilon$  to find equations that determine  $x_k$ :

$$\dot{x}_0 = f(t, x_0, 0), \quad x_0(t_0) = \eta_0$$

$$\dot{x}_1 = \frac{\partial f}{\partial x}(t, x_0(t), 0) x_1 + \frac{\partial f}{\partial \epsilon}(t, x_0(t), 0), \quad x_1(t_0) = \eta_1$$

Theorem 10.1. Suppose

- $f$  and its partial derivatives w.r.t  $(x, \epsilon)$  up to order  $N$  are continuous in  $(t, x, \epsilon)$  for  $(t, x, \epsilon) \in [t_0, t_1] \times D \times [-\epsilon_0, \epsilon_0]$  ( $f \in C^N$ )
- $\eta \in C^N$  for  $\epsilon \in [-\epsilon_0, \epsilon_0]$ .
- $\dot{x} = f(t, x, 0), x(t_0) = \eta_0$  has unique solution  $x_0(t)$  defined on  $[t_0, t_1]$  and  $x_0(t) \in D$  for all  $t \in [t_0, t_1]$ .

Then  $\exists \epsilon^* > 0$ , s.t.  $\forall |k| < \epsilon^*$ , the equation  $\dot{x} = f(t, x, \epsilon), x(t_0) = \eta(\epsilon)$  has a unique solution  $x(t, \epsilon)$  defined on  $[t_0, t_1]$  which satisfies:

$$x(t, \epsilon) - \sum_{k=0}^{N-1} x_k(t) \epsilon^k = O(\epsilon^N)$$

Perturbation on infinite interval.

Relax the assumption  $t \in [t_0, t_1]$  to  $t \in [t_0, 1/\epsilon]$  or  $[t_0, \infty]$ .

Look at the case when nominal system ( $\epsilon=0$ ) has e.s. eq. pt at 0.

Theorem 10.2.

Looks like assumptions of Theorem 10.1 with following modifications:

- $f$  & its partial derivatives w.r.t  $(x, \epsilon)$  up to order  $N$  are also bounded for any compact set  $D_0 \subset D$ .
- the origin is an e.s. eq. pt of nominal system

$$\dot{x}_0 = f(t, x_0, 0), \quad x_0(t_0) = \eta_0$$

with Lyapunov function  $V(t, x)$  that satisfies (Thm 4.9)

$$W_1(x) \leq V(t, x) \leq W_2(x)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, 0) \leq -W_3(x)$$

and  $\{W_1(x) \leq c\}$  is a compact subset of  $D$ .

Then for each compact set  $\Omega \subset \{W_2(x) \leq p_c, 0 < p_c < 1\}$ ,  $\exists \epsilon^* > 0$  s.t.

$\forall t_0 \geq 0$ ,  $\eta_0 \in \Omega$ , and  $|\epsilon| < \epsilon^*$

$$x(t, \epsilon) - \sum_{k=0}^{N-1} x_k(t) \epsilon^k = O(\epsilon^N) \quad \forall t \geq t_0$$

Periodic Perturbation  $\dot{x} = f(x) + \epsilon g(t, x, \epsilon)$

-  $f, g$ , and first partials w.r.t  $x$  are continuous and bounded for all

$(t, x, \epsilon) \in [0, \infty] \times D_0 \times [-\epsilon_0, \epsilon_0]$  for every compact set  $D_0 \subset D$  where

$0 \in D \subset \mathbb{R}^n$ .

Assume that the origin is e.s. eq. pt. of  $\dot{x} = f(x)$ .

Using result from chapter 9, all solutions of perturbed system approach  $D(\epsilon)$  neighborhood of origin as  $t \rightarrow \infty$  (ultimate boundedness).

Now suppose  $g$  is  $T$ -periodic,

$$g(t+T, x, \epsilon) = g(t, x, \epsilon), \quad \forall (t, x, \epsilon) \in [0, \infty] \times D \times [-\epsilon_0, \epsilon_0]$$

Is there a  $T$ -periodic solution inside  $O(\epsilon)$  neighborhood of origin?

**Theorem 10.3.** Under the previous assumptions,  $\exists \epsilon^* > 0$  and  $K > 0$  s.t.  $\forall \epsilon < \epsilon^*$ , Perturbed system has a unique  $T$ -periodic solution  $\bar{x}(t, \epsilon)$  with the property that  $\|\bar{x}(t, \epsilon)\| \leq K\epsilon$ . This solution is exponentially stable (in the sense of Lyapunov).

Further, if  $g(t, 0, \epsilon) = 0$ , the origin will be an eq. pt. of perturbed system. By uniqueness of  $\bar{x}(t, \epsilon)$ , we have that  $\bar{x}(t, \epsilon)$  is the trivial solution 0. I.e., the origin is e.s. for the perturbed system.

Proof uses Poincaré map of  $n+1$  dimensional system

$$\begin{cases} \dot{x} = f(x) + \epsilon g(t, x, \epsilon) \\ \dot{\theta} = 1 \end{cases}$$

Averaging  $\dot{x} = \epsilon f(t, x, \epsilon)$  (autonomous)

$0 < \epsilon \ll 1$  and  $f(t, x, \epsilon)$  is  $T$ -periodic in  $t$ .

Approximate system with averaged system which is average of  $f(t, x, 0)$ :

$$\dot{x} = \epsilon \bar{f}_{av}(x) \quad \bar{f}_{av}(x) = \frac{1}{T} \int_0^T f(t, x, 0) dt$$

(non-autonomous)

Idea If  $\epsilon$  is small, response of system is much slower than excitation. Thus, response governed primarily by average of excitation.

Average approximation error

Use change of variables to show that  $\dot{x} = \epsilon f(t, x, \epsilon)$  can be written as perturbation of  $\dot{x} = \epsilon \bar{f}_{av}(x)$ .

$$u(t, x) := \int_0^t h(\tau, x) d\tau ; \quad h(t, x) = f(t, x, 0) - f_{\text{av}}(x)$$

(f - its average)

change of variables:

$$x = y + \epsilon u(t, y)$$

$$\rightarrow \dot{x} = \dot{y} + \epsilon \frac{\partial u}{\partial t}(t, y) + \epsilon \frac{\partial u}{\partial y}(t, y) \dot{y}$$

$$\epsilon f(t, y + \epsilon u, \epsilon) = \left(I + \epsilon \frac{\partial u}{\partial y}\right) \dot{y} + \epsilon f(t, y, 0) - \epsilon f_{\text{av}}(y)$$

$$\Rightarrow \left(I + \epsilon \frac{\partial u}{\partial y}\right)^{-1} \dot{y} \stackrel{\text{def}}{=} \epsilon f_{\text{av}}(y) + \epsilon p(t, y, 0)$$

$$\text{where } p(t, y, 0) = \left[ f(t, y + \epsilon u, \epsilon) - f(t, y, 0) \right] + \underbrace{\left[ f(t, y, \epsilon) - f(t, y, 0) \right]}_{p(t, y, \epsilon)}$$

p: T-periodic in t, by mean value theorem,

$$p(t, y, \epsilon) = \frac{\partial f}{\partial x}(t, y + y^*, \epsilon) \in u + \frac{\partial f}{\partial \epsilon}(t, y, \epsilon^*) \epsilon$$

$0 < y^* < \epsilon u$        $0 < \epsilon^* < \epsilon$

$\frac{\partial u}{\partial y}$  : bounded on  $[0, \infty) \times D_0 \Rightarrow I + \epsilon \frac{\partial u}{\partial y}$  is non-singular and for

sufficiently small  $\epsilon$ ,  $\left(I + \epsilon \frac{\partial u}{\partial y}\right)^{-1} = I + O(\epsilon)$

Therefore,

$$\dot{y} = \epsilon f_{\text{av}}(y) + \epsilon^2 q(t, y, \epsilon)$$

$$\text{Let } s = \epsilon t. \text{ Then, } \boxed{\frac{dy}{ds} = f_{\text{av}}(y) + \epsilon q(s/\epsilon, y, \epsilon)}$$

where  $q(s/\epsilon, y, \epsilon)$  is ET-periodic in s & bounded on  $[0, \infty) \times D_0$  for sufficiently small  $\epsilon$ .

By chap 3 results, if averaged system has unique solution  $\bar{y}(s)$  on  $[0, b]$   $\bar{y}(s) \in D$ ,  $\forall s \in [0, b]$  and  $y(0, \epsilon) - \bar{y}_{\text{av}} = O(\epsilon)$ , then perturbed solution will have unique solution on  $[0, b]$  and the two will be  $O(\epsilon)$  close.

Since  $t = \frac{y}{\epsilon}$  and  $x - y = O(\epsilon)$ , then solutions of  $\dot{x} = \epsilon f_{av}(x)$  will be  $O(\epsilon)$  approximation of  $x(t, \epsilon)$ , the solution of  $\dot{x} = \epsilon f(t, x, \epsilon)$ , for  $t \in [0, b/\epsilon]$ .

### Stability

Suppose origin is e.s. eq. pt. of  $\dot{x} = f_{av}(x)$ . Let  $\Omega \subset D$  be a compact subset of the region of attraction. Suppose  $x_{av}(0) \in \Omega$  and  $x(0, \epsilon) - x_{av}(0) = O(\epsilon)$ . Then,  $\exists \epsilon^* > 0$  s.t.  $\forall 0 < \epsilon < \epsilon^*$ ,  $x(t, \epsilon)$  is defined and  $x(t, \epsilon) - x_{av}(\epsilon t) = O(\epsilon)$   $\forall t \in [0, \infty)$ .

Also, Original system will have unique, e.s., T-periodic solution  $\bar{x}(t, \epsilon)$  with  $\|\bar{x}(t, \epsilon)\| \leq k\epsilon$ .

